

# OBSERVABILITY ESTIMATE FOR STOCHASTIC SCHRÖDINGER EQUATIONS AND ITS APPLICATIONS\*

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**Abstract.** In this paper, we establish a boundary observability estimate for stochastic Schrödinger equations by means of the global Carleman estimate. Our Carleman estimate is based on a new fundamental identity for a stochastic Schrödinger-like operator. Applications to the state observation problem for semilinear stochastic Schrödinger equations and the unique continuation problem for stochastic Schrödinger equations are also addressed.

**Key words.** stochastic Schrödinger equation, global Carleman estimate, observability estimate, state observation problem, unique continuation property

**AMS subject classifications.** 93B07, 35B45

**1. Introduction and Main Results.** Let  $T > 0$ ,  $G \subset \mathbb{R}^n$  ( $n \in \mathbb{N}$ ) be a given bounded domain with a  $C^2$  boundary  $\Gamma$ . Let  $\Gamma_0$  be a suitable chosen nonempty subset (to be given later) of  $\Gamma$ . Put  $Q \triangleq (0, T) \times G$ ,  $\Sigma \triangleq (0, T) \times \Gamma$ , and  $\Sigma_0 \triangleq (0, T) \times \Gamma_0$ .

Let  $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, P)$  be a complete filtered probability space on which a one dimensional standard Brownian motion  $\{B(t)\}_{t \geq 0}$  is defined. Let  $H$  be a Banach space. Denote by  $L^2_{\mathcal{F}}(0, T; H)$  the Banach space consisting of all  $H$ -valued  $\{\mathcal{F}_t\}_{t \geq 0}$ -adapted processes  $X(\cdot)$  such that  $\mathbb{E}(|X(\cdot)|^2_{L^2(0, T; H)}) < \infty$ ; by  $L^\infty_{\mathcal{F}}(0, T; H)$  the Banach space consisting of all  $H$ -valued  $\{\mathcal{F}_t\}_{t \geq 0}$ -adapted bounded processes; and by  $L^2_{\mathcal{F}}(\Omega; C([0, T]; H))$  the Banach space consisting of all  $H$ -valued  $\{\mathcal{F}_t\}_{t \geq 0}$ -adapted processes  $X(\cdot)$  such that  $\mathbb{E}(|X(\cdot)|^2_{C([0, T]; H)}) < \infty$ . All of these spaces are endowed with the canonical norm. Put

$$H_T \triangleq L^2_{\mathcal{F}}(\Omega; C([0, T]; H_0^1(G))).$$

Let us consider the following stochastic Schrödinger equation:

$$\begin{cases} idy + \Delta y dt = (a_1 \cdot \nabla y + a_2 y + f)dt + (a_3 y + g)dB(t) & \text{in } Q, \\ y = 0 & \text{on } \Sigma, \\ y(0) = y_0 & \text{in } G, \end{cases} \quad (1.1)$$

with initial datum  $y_0 \in L^2(\Omega, \mathcal{F}_0, P; H_0^1(G))$ , suitable coefficients  $a_i$  ( $i = 1, 2, 3$ ), and source terms  $f$  and  $g$ . The solution to (1.1) is understood in the following sense.

**DEFINITION 1.1.** *We call  $y \in H_T$  a solution to the equation (1.1) if*

1.  $y(0) = y_0$  in  $G$ ,  $P$ -a.s.;

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\*This work is partially supported by the NSF of China under grant 11101070, and the ERC Advanced Grant FP7-246775 NUMERIWAVES, the Grant PI2010-04 of the Basque Government, the ESF Research Networking Programme OPTPDE and Grant MTM2008-03541 of the MICINN, Spain.

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2. For any  $t \in [0, T]$  and  $\eta \in H_0^1(G)$ , it holds that

$$\begin{aligned} & \int_G iy(t, x)\eta(x)dx - \int_G iy(0, x)\eta(x)dx \\ &= \int_0^t \int_G \left[ \nabla y(s, x) \cdot \nabla \eta(x) + (a_1 \cdot \nabla y + a_2 y + f)\eta(x) \right] dx ds \\ &+ \int_0^t \int_G (a_3 y + g)\eta(x) dx dB(s), \quad P\text{-a.s.} \end{aligned}$$

We refer to [5, Chapter 6] for the well-posedness of the equation (1.1) in  $H_T$ , under suitable assumptions (the assumptions in this paper are enough).

Similar to its deterministic counterpart, the stochastic Schrödinger equation plays an important role in quantum mechanics. We refer the readers to [2, 13] and the rich references therein for the details of its physical background.

The main purpose of this paper is to establish a boundary observability estimate for the equation (1.1) in the following setting.

Denote by  $\nu(x)$  the unit outward normal vector of  $G$  at  $x \in \Gamma$ . Let  $x_0 \in (\mathbb{R}^n \setminus \overline{G})$ . In what follows, we choose

$$\Gamma_0 = \{x \in \Gamma : (x - x_0) \cdot \nu(x) > 0\}. \quad (1.2)$$

We assume that

$$\begin{cases} ia_1 \in L_{\mathcal{F}}^\infty(0, T; W_0^{1,\infty}(G; \mathbb{R}^n)), \\ a_2 \in L_{\mathcal{F}}^\infty(0, T; W^{1,\infty}(G)), \\ a_3 \in L_{\mathcal{F}}^\infty(0, T; W^{1,\infty}(G)), \end{cases} \quad (1.3)$$

and that

$$\begin{cases} f \in L_{\mathcal{F}}^2(0, T; H_0^1(G)), \\ g \in L_{\mathcal{F}}^2(0, T; H^1(G)). \end{cases} \quad (1.4)$$

In the sequel, we put

$$r_1 \triangleq |a_1|_{L_{\mathcal{F}}^\infty(0, T; W_0^{1,\infty}(G; \mathbb{R}^n))}^2 + |a_2|_{L_{\mathcal{F}}^\infty(0, T; W^{1,\infty}(G))}^2 + |a_3|_{L_{\mathcal{F}}^\infty(0, T; W^{1,\infty}(G))}^2 + 1, \quad (1.5)$$

and denote by  $C$  a generic positive constant depending only on  $T$ ,  $G$  and  $x_0$ , which may change from line to line.

Now we state the main result of this paper as follows.

**THEOREM 1.2.** *If the conditions (1.2)–(1.4) hold, then any solution of the equation (1.1) satisfies that*

$$\begin{aligned} & |y_0|_{L^2(\Omega, \mathcal{F}_0, P; H_0^1(G))} \\ & \leq e^{Cr_1} \left( \left| \frac{\partial y}{\partial \nu} \right|_{L_{\mathcal{F}}^2(0, T; L^2(\Gamma_0))} + |f|_{L_{\mathcal{F}}^2(0, T; H_0^1(G))} + |g|_{L_{\mathcal{F}}^2(0, T; H^1(G))} \right). \end{aligned} \quad (1.6)$$

**REMARK 1.1.** *Since  $y$  belongs only to  $H_T$ , its normal derivative  $\frac{\partial y}{\partial \nu}$  may not make sense. Fortunately, due to the hidden regularity of the solution to the equation*

(1.1), one can show that  $\frac{\partial y}{\partial \nu}$  exists and belongs to  $L^2_{\mathcal{F}}(0, T; L^2(\Gamma))$  (see Proposition 2.2 for more details).

It is well-known that observability estimates (in the spirit of (1.6)) for partial differential equations play fundamental role in proving the controllability of the dual control systems. There exist many approaches and results addressing the observability estimate for deterministic Schrödinger equations. For example, similar results in the spirit of Theorem 1.2 are obtained by Carleman estimate (e.g. [3, 15, 22]), by the classical Rellich-type multiplier approach ([21]), by the microlocal analysis approach ([16, 23]), and so on. We refer to [32] for a nice survey in this respect. However, people know very little about the stochastic counterpart. To our best knowledge, [19] is the only published result for this problem, where partial results in this paper have been announced without detailed proofs.

Besides its important application to the controllability problem, the observability estimate not only have its own interest (a kind of energy estimate and quantitative uniqueness for the solution) but also has some other important applications. For instance, a typical application of this sort of estimates is to study the state observation problem, that is, to determine the state of a system by a suitable observation. Once the observability is obtained, we may conclude that the state can be uniquely determined from the observed data and continuously depends on it. For instance, once the inequality (1.6) is established, it follows that  $y \in H_T$  is determined by  $\frac{\partial y}{\partial \nu}|_{(0,T) \times \Gamma_0}$  continuously. In Section 6, we shall consider a state observation problem for semilinear stochastic Schrödinger equations.

In this paper, we will prove Theorem 1.2 by applying the global Carleman estimate (See Theorem 1.3 below).

We now introduce the weight functions to be used in our Carleman estimate. Let

$$\psi(x) = |x - x_0|^2 + \tau, \quad (1.7)$$

where  $\tau$  is a positive constant such that  $\psi \geq \frac{5}{6}|\psi|_{L^\infty(G)}$ . Let  $s > 0$  and  $\lambda > 0$ . Put

$$\ell = s \frac{e^{4\lambda\psi} - e^{5\lambda|\psi|_{L^\infty(G)}}}{t^2(T-t)^2}, \quad \varphi = \frac{e^{4\lambda\psi}}{t^2(T-t)^2}, \quad \theta = e^\ell. \quad (1.8)$$

We have the following global Carleman inequality.

**THEOREM 1.3.** *According to (1.2)–(1.5) and (1.8), there is an  $s_1 > 0$  (depending on  $r_1$ ) and a  $\lambda_1 > 0$  such that for each  $s \geq s_1$ ,  $\lambda \geq \lambda_1$  and for any solution of the equation (1.1), it holds that*

$$\begin{aligned} & \mathbb{E} \int_Q \theta^2 \left( s^3 \lambda^4 \varphi^3 |y|^2 + s \lambda \varphi |\nabla y|^2 \right) dx dt \\ & \leq C \left\{ \mathbb{E} \int_Q \theta^2 \left( |f|^2 + s^2 \lambda^2 \varphi^2 |g|^2 + |\nabla g|^2 \right) dx dt + \mathbb{E} \int_0^T \int_{\Gamma_0} \theta^2 s \lambda \varphi \left| \frac{\partial y}{\partial \nu} \right|^2 d\Gamma dt \right\}. \end{aligned} \quad (1.9)$$

Further, if  $g \in L^2_{\mathcal{F}}(0, T; H^1(G; \mathbb{R}))$ , then (1.9) can be strengthened as the following:

$$\begin{aligned} & \mathbb{E} \int_Q \theta^2 \left( s^3 \lambda^4 \varphi^3 |y|^2 + s \lambda \varphi |\nabla y|^2 \right) dx dt \\ & \leq C \left\{ \mathbb{E} \int_Q \theta^2 \left( |f|^2 + s^2 \lambda^2 \varphi^2 |g|^2 \right) dx dt + \mathbb{E} \int_0^T \int_{\Gamma_0} \theta^2 s \lambda \varphi \left| \frac{\partial y}{\partial \nu} \right|^2 d\Gamma dt \right\}. \end{aligned} \quad (1.10)$$

Carleman estimate is an important tool for the study of unique continuation property, stabilization, controllability and inverse problems for deterministic partial differential equations (e.g. [3, 15, 22, 25, 26, 32]). Although there are numerous results for the Carleman estimate for deterministic partial differential equations, people know very little about the corresponding stochastic situation. In fact, as far as we know, [1, 19, 20, 24, 30] are the only five published papers addressing the Carleman estimate for stochastic partial differential equations. The references [1, 20, 24] are devoted to stochastic heat equations, while [30] is concerned with stochastic wave equations. In [19], Theorem 1.3 was announced without proof.

At first glance, the proof of Theorem 1.3 looks very similar to that of the global Carleman estimate for (stochastic) parabolic equations (See [10, 24]). Furthermore, one can find that the idea behind the proofs in this paper and [10, 24] are analogous. Nevertheless, the specific proofs have big differences. First, we have to choose different weight functions. Second, we deal with different equations. Such kind of differences lead to considerably different difficulties in the proof of Theorem 1.3. One cannot simply mimic the proofs in [10, 24] to obtain Theorem 1.3. Indeed, even in the deterministic setting, the proof of the global Carleman estimate for Schrödinger equations are much more complicated than that for the parabolic and hyperbolic equations (see [27, 15]).

The rest of this paper is organized as follows. In Section 2, we give some preliminary results, including an energy estimate and the hidden regularity for solutions of the equation (1.1). Section 3 is addressed to establish a crucial identity for a stochastic Schrödinger-like operator. Then, in Section 4, we derive the desired Carleman estimate. Section 5 is devoted to prove Theorem 1.2. In Section 6, as applications of the observability/Carleman estimates developed in this work, we study a state observation problem for semilinear stochastic Schrödinger equations and establish a unique continuation property for the solution to the equation (1.1). Finally, we present some further comments and open problems concerned with this paper in Section 7.

**2. Some preliminaries.** In this section, we give some preliminary results which will be used later.

To begin with, for the sake of completeness, we give an energy estimate for the equation (1.1).

**PROPOSITION 2.1.** *According to (1.2)–(1.5), for all  $y$  which solve the equation (1.1), it holds that*

$$\mathbb{E}|y(t)|_{H_0^1(G)}^2 \leq e^{Cr_1} \left( \mathbb{E}|y(s)|_{H_0^1(G)}^2 + |f|_{L_{\mathcal{F}}^2(0,T;H_0^1(G))}^2 + |g|_{L_{\mathcal{F}}^2(0,T;H_0^1(G))}^2 \right), \quad (2.1)$$

for any  $s, t \in [0, T]$ .

*Proof :* Without loss of generality, we assume that  $t < s$ . To begin with, we compute  $\mathbb{E}|y(t)|_{L^2(G)}^2 - \mathbb{E}|y(s)|_{L^2(G)}^2$  and  $\mathbb{E}|\nabla y(t)|_{L^2(G)}^2 - \mathbb{E}|\nabla y(s)|_{L^2(G)}^2$ . The first one

reads

$$\begin{aligned}
 & \mathbb{E}|y(t)|_{L^2(G)}^2 - \mathbb{E}|y(s)|_{L^2(G)}^2 \\
 &= -\mathbb{E} \int_t^s \int_G (y d\bar{y} + \bar{y} dy + dy d\bar{y}) dx \\
 &= \mathbb{E} \int_t^s \int_G \left\{ i y (\Delta \bar{y} - a_1 \cdot \nabla \bar{y} - a_2 \bar{y} - \bar{f}) - i \bar{y} (\Delta y - a_1 \cdot \nabla y - a_2 y - f) \right. \\
 &\quad \left. - (a_3 y + g)(a_3 \bar{y} + \bar{g}) \right\} dx d\sigma \\
 &= \mathbb{E} \int_t^s \int_G \left\{ i [\operatorname{div}(y \nabla \bar{y}) - |\nabla y|^2 - \operatorname{div}(|y|^2 a_1) + \operatorname{div}(a_1)|y|^2 - a_2|y|^2 - y \bar{f}] \right. \\
 &\quad \left. - i [\operatorname{div}(\bar{y} \nabla y) - |\nabla \bar{y}|^2 - \operatorname{div}(|\bar{y}|^2 a_1) + \operatorname{div}(a_1)|\bar{y}|^2 - a_2|\bar{y}|^2 - f \bar{y}] \right. \\
 &\quad \left. - (a_3 y + g)(a_3 \bar{y} + \bar{g}) \right\} dx d\sigma \\
 &\leq \mathbb{E} \int_t^s 2 \left[ (|a_3|_{L^\infty(G)} + 1) |y|_{L^2(G)}^2 + |f|_{L^2(G)}^2 + |g|_{L^2(G)}^2 \right] dx d\sigma.
 \end{aligned} \tag{2.2}$$

The second one is

$$\begin{aligned}
 & \mathbb{E}|\nabla y(t)|_{L^2(G)}^2 - \mathbb{E}|\nabla y(s)|_{L^2(G)}^2 \\
 &= -\mathbb{E} \int_t^s \int_G (\nabla y d\nabla \bar{y} + \nabla \bar{y} d\nabla y + d\nabla y d\nabla \bar{y}) dx \\
 &= -\mathbb{E} \int_t^s \int_G \left\{ \operatorname{div}(\nabla y d\bar{y}) - \Delta y d\bar{y} + \operatorname{div}(\nabla \bar{y} dy) - \Delta \bar{y} dy + d\nabla y d\nabla \bar{y} \right\} dx \\
 &= -\mathbb{E} \int_t^s \int_G \left\{ \Delta y \left[ i(\Delta \bar{y} - a_1 \cdot \nabla \bar{y} - a_2 \bar{y} - \bar{f}) \right] - \Delta \bar{y} \left[ i(\Delta y - a_1 \cdot \nabla y - a_2 y - f) \right] \right. \\
 &\quad \left. + \nabla(a_3 y + g) \nabla(a_3 \bar{y} + \bar{g}) \right\} dx d\sigma \\
 &\leq 2\mathbb{E} \int_t^s \left\{ (|a_1|_{W^{1,\infty}(G;\mathbb{R}^m)}^2 + |a_3|_{W^{1,\infty}(G)}^2 + 1) |\nabla y|_{L^2(G)}^2 \right. \\
 &\quad \left. + (|a_2|_{W^{1,\infty}(G)}^2 + |a_3|_{W^{1,\infty}(G)}^2 + 1) |y|_{L^2(G)}^2 + |f|_{H_0^1(G)}^2 + |g|_{H_0^1(G)}^2 \right\} dx d\sigma.
 \end{aligned} \tag{2.3}$$

From (2.2) and (2.3), we have that

$$\begin{aligned}
 & \mathbb{E}|y(t)|_{H_0^1(G)}^2 - \mathbb{E}|y(s)|_{H_0^1(G)}^2 \\
 &\leq 2(r_1 + 1) \mathbb{E} \int_t^s \int_G |y(\sigma)|_{H_0^1(G)}^2 dx d\sigma + \mathbb{E} \int_t^s \int_G (|f(\sigma)|_{H_0^1(G)}^2 + |g(\sigma)|_{H_0^1(G)}^2) dx d\sigma.
 \end{aligned} \tag{2.4}$$

From this, and thanks to Gronwall's inequality, we arrive at

$$\mathbb{E}|y(t)|_{H_0^1(G)}^2 \leq e^{2(r_1+1)} \left\{ \mathbb{E}|y(s)|_{H_0^1(G)}^2 + \mathbb{E} \int_0^T \int_G (|f|_{H_0^1(G)}^2 + |g|_{H_0^1(G)}^2) dx d\sigma \right\}, \tag{2.5}$$

which implies the inequality (2.1) immediately.  $\square$

REMARK 2.1. *The proof of this proposition is almost standard. However, people may doubt the correctness of the inequality (2.1) for  $t < s$  because of the very fact*

that the equation (1.1) is time irreversible. Fortunately, the inequality (2.1) is true for  $t < s$ . In fact, in the stochastic setting one should divide the time irreversible systems into two classes. The first class of time irreversibility is caused by the energy dissipation. Thus, one cannot estimate the energy of the system at time  $t$  by that at time  $s$  uniformly when  $t < s$ . A typical example of such kind of systems is the heat equation. The second class of time irreversibility comes from the stochastic noise. Such kind of system cannot be solved backward, that is, if we give the final data rather than the initial data, then the system is not well-posed (Recall that, this is the very starting point of backward stochastic differential equations). Stochastic Schrödinger equations and stochastic wave equations are typical systems of the second class. For these systems, we can still estimate the energy at time  $t$  by that at time  $s$  for  $t < s$ .

Next, we give a result concerning the hidden regularity for solutions of the equation (1.1). It shows that, solutions of this equation enjoy a higher regularity on the boundary than the one provided by the classical trace theorem for Sobolev spaces.

**PROPOSITION 2.2.** *According to (1.2)–(1.5), for any solution of the equation (1.1), it holds that*

$$\begin{aligned} & \left| \frac{\partial y}{\partial \nu} \right|_{L^2_{\mathcal{F}}(0,T;L^2(\Gamma_0))}^2 \\ & \leq e^{Cr_1} \left( |y_0|_{L^2(\Omega, \mathcal{F}_0, P; H_0^1(G))}^2 + |f|_{L^2_{\mathcal{F}}(0,T;H_0^1(G))}^2 + |g|_{L^2_{\mathcal{F}}(0,T;H^1(G))}^2 \right). \end{aligned} \quad (2.6)$$

**REMARK 2.2.** *By means of Proposition 2.2, we know that  $\left| \frac{\partial y}{\partial \nu} \right|_{L^2_{\mathcal{F}}(0,T;L^2(\Gamma_0))}^2$  makes sense. Compared with Theorem 1.2, Proposition 2.2 tells us the fact that  $\left| \frac{\partial y}{\partial \nu} \right|_{L^2_{\mathcal{F}}(0,T;L^2(\Gamma_0))}^2$  can be bounded by the initial datum and non-homogenous terms. This result is the converse of Theorem 1.2 in some sense.*

To prove Proposition 2.2, we first establish a pointwise identity. For simplicity, here and in the sequel, we adopt the notation  $y_i \equiv y_i(x) \triangleq \frac{\partial y(x)}{\partial x_i}$ , where  $x_i$  is the  $i$ -th coordinate of a generic point  $x = (x_1, \dots, x_n)$  in  $\mathbb{R}^n$ . In a similar manner, we use the notation  $z_i, v_i$ , etc., for the partial derivatives of  $z$  and  $v$  with respect to  $x_i$ .

**PROPOSITION 2.3.** *Let  $\mu = \mu(x) = (\mu^1, \dots, \mu^n) : \mathbb{R}^n \rightarrow \mathbb{R}^n$  be a vector field of class  $C^1$  and  $z$  an  $H_{loc}^2(\mathbb{R}^n)$ -valued  $\{\mathcal{F}_t\}_{t \geq 0}$ -adapted process. Then for a.e.  $x \in \mathbb{R}^n$  and  $P$ -a.s.  $\omega \in \Omega$ , it holds that*

$$\begin{aligned} & \mu \cdot \nabla \bar{z}(idz + \Delta z dt) + \mu \cdot \nabla z(-id\bar{z} + \Delta \bar{z} dt) \\ & = \nabla \cdot \left[ (\mu \cdot \nabla \bar{z}) \nabla z + (\mu \cdot \nabla z) \nabla \bar{z} - i(z d\bar{z}) \mu - |\nabla z|^2 \mu \right] dt + d(i\mu \cdot \nabla \bar{z} z) \\ & \quad - 2 \sum_{j,k=1}^n \mu_j^k z_j \bar{z}_k dt + (\nabla \cdot \mu) |\nabla z|^2 dt + i(\nabla \cdot \mu) z d\bar{z} - i(\mu \cdot \nabla d\bar{z}) dz. \end{aligned} \quad (2.7)$$

*Proof of Proposition 2.3 :* The proof is a direct computation. We have that

$$\begin{aligned} & \sum_{k=1}^n \sum_{j=1}^n \mu^k \bar{z}_k z_{jj} + \sum_{k=1}^n \sum_{j=1}^n \mu^k z_k \bar{z}_{jj} \\ &= \sum_{k=1}^n \sum_{j=1}^n \left[ (\mu^k \bar{z}_k z_j)_j + (\mu^k z_k \bar{z}_j)_j + \mu_k^k |z_j|^2 - (\mu^k |z_j|^2)_k - 2\mu_j^k \bar{z}_k z_j \right] \end{aligned} \quad (2.8)$$

and that

$$\begin{aligned} & i \sum_{k=1}^n (\mu^k \bar{z}_k dz - \mu^k z_k d\bar{z}) \\ &= i \sum_{k=1}^n \left[ d(\mu^k \bar{z}_k z) - \mu^k z d\bar{z}_k - \mu^k d\bar{z}_k z - (\mu^k z d\bar{z})_k + \mu^k z d\bar{z}_k + \mu_k^k z d\bar{z} \right] \\ &= i \sum_{k=1}^n \left[ d(\mu^k \bar{z}_k z) - \mu^k d\bar{z}_k z - (\mu^k z d\bar{z})_k + \mu_k^k z d\bar{z} \right]. \end{aligned} \quad (2.9)$$

Combining (2.8) and (2.9), we get the equality (2.7).  $\square$

By virtue of Proposition 2.3, the proof of Proposition 2.2 is standard. We only give a sketch here.

*Sketch of the Proof of Proposition 2.2 :* Since  $\Gamma$  is  $C^2$ , one can find a vector field  $\mu_0 = (\mu_0^1, \dots, \mu_0^n) \in C^1(\bar{G}; \mathbb{R}^n)$  such that  $\mu_0 = \nu$  on  $\Gamma$  (see [14, page 18] for the construction of  $\mu_0$ ). Letting  $\mu = \mu_0$  and  $z = y$  in Proposition 2.3, integrating it in  $Q$  and taking the expectation, by means of Proposition 2.3, with similar computation in [26], Proposition 2.2 can be obtained immediately.

**3. An Identity for a Stochastic Schrödinger-like Operator.** In this section, we obtain an identity for a stochastic schrödinger-like operator, which is similar to the formula (2.7) in the spirit but it takes a more complex form and play a key role in the proof of Theorem 1.3.

Let  $\beta(t, x) \in C^2(\mathbb{R}^{1+n}; \mathbb{R})$ , and let  $b^{jk}(t, x) \in C^{1,2}(\mathbb{R}^{1+n}; \mathbb{R})$  satisfy

$$b^{jk} = b^{kj}, \quad j, k = 1, 2, \dots, n. \quad (3.1)$$

Let us define a (formal) second order stochastic partial differential operator  $\mathcal{P}$  as

$$\mathcal{P}z \triangleq i\beta(t, x)dz + \sum_{j,k=1}^n (b^{jk}(t, x)z_j)_k dt, \quad i = \sqrt{-1}. \quad (3.2)$$

We have the following equality concerning  $\mathcal{P}$ :

**THEOREM 3.1.** *Let  $\ell, \Psi \in C^2(\mathbb{R}^{1+n}; \mathbb{R})$ . Assume that  $z$  is an  $H_{loc}^2(\mathbb{R}^n, \mathbb{C})$ -valued  $\{\mathcal{F}_t\}_{t \geq 0}$ -adapted process. Put  $v = \theta z$  (recall (1.8) for the definition of  $\theta$ ). Then for a.e.  $x \in \mathbb{R}^n$  and  $P$ -a.s.  $\omega \in \Omega$ , it holds that*

$$\begin{aligned}
& \theta(\mathcal{P}z\overline{I_1} + \overline{\mathcal{P}z}I_1) + dM + \operatorname{div} V \\
= & 2|I_1|^2 dt + \sum_{j,k=1}^n c^{jk}(v_k \overline{v_j} + \overline{v_k} v_j) dt + D|v|^2 dt \\
& + i \sum_{j,k=1}^n \left[ (\beta b^{jk} \ell_j)_t + b^{jk}(\beta \ell_t)_j \right] (\overline{v_k} v - v_k \overline{v}) dt \\
& + i \left[ \beta \Psi + \sum_{j,k=1}^n (\beta b^{jk} \ell_j)_k \right] (\overline{v} dv - v d\overline{v}) \\
& + (\beta^2 \ell_t) dv d\overline{v} + i \sum_{j,k=1}^n \beta b^{jk} \ell_j (dv d\overline{v}_k - dv_k d\overline{v}),
\end{aligned} \tag{3.3}$$

where

$$\begin{cases} I_1 \triangleq -i\beta \ell_t v - 2 \sum_{j,k=1}^n b^{jk} \ell_j v_k + \Psi v, \\ A \triangleq \sum_{j,k=1}^n b^{jk} \ell_j \ell_k - \sum_{j,k=1}^n (b^{jk} \ell_j)_k - \Psi, \end{cases} \tag{3.4}$$

$$\begin{cases} M \triangleq \beta^2 \ell_t |v|^2 + i\beta \sum_{j,k=1}^n b^{jk} \ell_j (\overline{v_k} v - v_k \overline{v}), \\ V \triangleq [V^1, \dots, V^k, \dots, V^n], \\ V^k \triangleq -i\beta \sum_{j=1}^n \left[ b^{jk} \ell_j (v d\overline{v} - \overline{v} dv) + b^{jk} \ell_t (v_j \overline{v} - \overline{v_j} v) dt \right] \\ \quad - \Psi \sum_{j=1}^n b^{jk} (v_j \overline{v} + \overline{v_j} v) dt + \sum_{j=1}^n b^{jk} (2A \ell_j + \Psi_j) |v|^2 dt \\ \quad + \sum_{j,j',k'=1}^n \left( 2b^{jk'} b^{j'k} - b^{jk} b^{j'k'} \right) \ell_j (v_{j'} \overline{v_{k'}} + \overline{v_{j'}} v_{k'}) dt, \end{cases} \tag{3.5}$$

and

$$\begin{cases} c^{jk} \triangleq \sum_{j',k'=1}^n \left[ 2(b^{j'k} \ell_{j'})_{k'} b^{j'k'} - (b^{jk} b^{j'k'} \ell_{j'})_{k'} \right] - b^{jk} \Psi, \\ D \triangleq (\beta^2 \ell_t)_t + \sum_{j,k=1}^n (b^{jk} \Psi_k)_j + 2 \left[ \sum_{j,k=1}^n (b^{jk} \ell_j A)_k + A \Psi \right]. \end{cases} \tag{3.6}$$

REMARK 3.1. Since we only assume that  $(b^{jk})_{1 \leq j,k \leq n}$  is symmetric and do not assume that it is positively definite, then similar to [7] and based on the identity



(3.3) in Theorem 3.1, we can deduce observability estimate not only for the stochastic Schrödinger equation, but also for deterministic hyperbolic, Schrödinger and plate equations, which had been derived via Carleman estimate (see [9], [15] and [27], respectively).

*Proof of Theorem 3.1:* The proof is divided into three steps.

**Step 1.** By the definition of  $v$  and  $w$ , a straightforward computation shows that:

$$\begin{aligned} \theta \mathcal{P}z &= i\beta dv - i\beta \ell_t v dt + \sum_{j,k=1}^n (b^{jk} v_j)_k dt \\ &\quad + \sum_{j,k=1}^n b^{jk} \ell_j \ell_k v dt - 2 \sum_{j,k=1}^n b^{jk} \ell_j v_k dt - \sum_{j,k=1}^n (b^{jk} \ell_j)_k v dt \\ &= I_1 dt + I_2, \end{aligned} \quad (3.7)$$

where

$$I_2 = i\beta dv + \sum_{j,k=1}^n (b^{jk} v_j)_k dt + A v dt. \quad (3.8)$$

Hence we obtain that

$$\theta(Pz\overline{I_1} + \overline{Pz}I_1) = 2|I_1|^2 dt + (I_1\overline{I_2} + I_2\overline{I_1}). \quad (3.9)$$

**Step 2.** In this step, we compute  $I_1\overline{I_2} + I_2\overline{I_1}$ . Denote the three terms in  $I_1$  and  $I_2$  by  $I_1^j$  and  $I_2^j$ ,  $j = 1, 2, 3$ , respectively. Then we have that

$$\begin{aligned} &I_1^1\overline{I_2^1} + I_2^1\overline{I_1^1} \\ &= -i\beta \ell_t v \overline{(i\beta dv)} + i\beta dv \overline{(-i\beta \ell_t v)} \\ &= -d(\beta^2 \ell_t |v|^2) + (\beta^2 \ell_t)_t |v|^2 dt + \beta^2 \ell_t dv d\overline{v}. \end{aligned} \quad (3.10)$$

Noting that

$$\begin{cases} 2vd\overline{v} = d(|v|^2) - (\overline{v}dv - v d\overline{v}) - dv d\overline{v}, \\ 2v\overline{v}_k = (|v|^2)_k - (\overline{v}v_k - v\overline{v}_k), \end{cases} \quad (3.11)$$

we find first

$$\begin{aligned} &2i \sum_{j,k=1}^n (\beta b^{jk} \ell_j v d\overline{v})_k \\ &= i \sum_{j,k=1}^n \left\{ \beta b^{jk} \ell_j \left[ d(|v|^2) - (\overline{v}dv - v d\overline{v}) - dv d\overline{v} \right] \right\}_k \\ &= i \sum_{j,k=1}^n \left\{ (\beta b^{jk} \ell_j)_k d(|v|^2) + \beta b^{jk} \ell_j [d(|v|^2)]_k - [\beta b^{jk} \ell_j (\overline{v}dv - v d\overline{v})]_k \right. \\ &\quad \left. - (\beta b^{jk} \ell_j)_k dv d\overline{v} - \beta b^{jk} \ell_j dv_k d\overline{v} - \beta b^{jk} \ell_j dv d\overline{v}_k \right\}, \end{aligned} \quad (3.12)$$

next

$$\begin{aligned}
& -2i \sum_{j,k=1}^n (\beta b^{jk} \ell_j)_k v d\bar{v} \\
& = -i \sum_{j,k=1}^n (\beta b^{jk} \ell_j)_k \left[ d(|v|^2) - (\bar{v} dv - v d\bar{v}) - dv d\bar{v} \right] \\
& = -i \sum_{j,k=1}^n \left[ (\beta b^{jk} \ell_j)_k d(|v|^2) - (\beta b^{jk} \ell_j)_k (\bar{v} dv - v d\bar{v}) - (\beta b^{jk} \ell_j)_k dv d\bar{v} \right],
\end{aligned} \tag{3.13}$$

then

$$\begin{aligned}
& -2i \sum_{j,k=1}^n d(\beta b^{jk} \ell_j v \bar{v}_k) \\
& = -i \sum_{j,k=1}^n d \left\{ \beta b^{jk} \ell_j [(|v|^2)_k - (\bar{v} v_k - v \bar{v}_k)] \right\} \\
& = -i \sum_{j,k=1}^n \left\{ (\beta b^{jk} \ell_j)_t (|v|^2)_k dt + \beta b^{jk} \ell_j d[(|v|^2)_k] - d[\beta b^{jk} \ell_j (\bar{v} v_k - v \bar{v}_k)] \right\},
\end{aligned} \tag{3.14}$$

and that

$$\begin{aligned}
& 2i \sum_{j,k=1}^n (\beta b^{jk} \ell_j)_t v \bar{v}_k dt \\
& = i \sum_{j,k=1}^n d(\beta b^{jk} \ell_j)_t [(|v|^2)_k - (\bar{v} v_k - v \bar{v}_k)] dt \\
& = i \left[ \sum_{j,k=1}^n (\beta b^{jk} \ell_j)_t (|v|^2)_k dt - (\beta b^{jk} \ell_j)_t (\bar{v} v_k - v \bar{v}_k) dt \right].
\end{aligned} \tag{3.15}$$

From (3.12)–(3.15), we get that

$$\begin{aligned}
& (I_1^2 + I_1^3) \bar{I}_2^1 + I_2^1 (\bar{I}_1^2 + \bar{I}_1^3) \\
& = \left( -2 \sum_{j,k=1}^n b^{jk} \ell_j v_k + \Psi v \right) \overline{(i\beta dv)} + i\beta dv \overline{\left( -2 \sum_{j,k=1}^n b^{jk} \ell_j v_k + \Psi v \right)} \\
& = 2i \sum_{j,k=1}^n \beta b^{jk} \ell_j (v_k d\bar{v} - \bar{v}_k dv) + i\beta \Psi (\bar{v} dv - v d\bar{v}) \\
& = 2i \sum_{j,k=1}^n \left[ (\beta b^{jk} \ell_j v d\bar{v})_k - (\beta b^{jk} \ell_j)_k v d\bar{v} - \beta b^{jk} \ell_j v d\bar{v}_k \right] \\
& \quad - 2i \sum_{j,k=1}^n \left[ d(\beta b^{jk} \ell_j v \bar{v}_k) - (\beta b^{jk} \ell_j)_t v \bar{v}_k dt - \beta b^{jk} \ell_j v d\bar{v}_k \right]
\end{aligned} \tag{3.16}$$

$$\begin{aligned}
& +2i \sum_{j,k=1}^n \beta b^{jk} \ell_j dv d\bar{v}_k + i\beta \Psi(\bar{v}dv - vd\bar{v}) \\
& = -i \sum_{j,k=1}^n \left[ \beta b^{jk} \ell_j (\bar{v}dv - vd\bar{v}) \right]_k dt - i \sum_{j,k=1}^n d \left[ \beta b^{jk} \ell_j (v\bar{v}_k - \bar{v}v_k) \right] \\
& \quad -i \sum_{j,k=1}^n (\beta b^{jk} \ell_j)_t (\bar{v}v_k - v\bar{v}_k) dt + i \left[ \beta \Psi + \sum_{j,k=1}^n (\beta b^{jk} \ell_j)_k \right] (\bar{v}dv - vd\bar{v}) \\
& \quad +i \sum_{j,k=1}^n \beta b^{jk} \ell_j (dv d\bar{v}_k - dv_k d\bar{v}).
\end{aligned}$$

Noting that  $b^{jk} = b^{kj}$ , we have that

$$\begin{aligned}
& I_1^1 \overline{I_2^2} + I_2^2 \overline{I_1^1} \\
& = -i\beta \ell_t v \overline{\sum_{j,k=1}^n (b^{jk} v_j)_k dt} + \sum_{j,k=1}^n (b^{jk} v_j)_k \overline{(-i\beta \ell_t v)} \\
& = \sum_{j,k=1}^n \left[ i\beta b^{jk} \ell_t (v_j \bar{v} - \bar{v}_j v) \right]_k dt + i \sum_{j,k=1}^n b^{jk} (\beta \ell_t)_k (\bar{v}_j v - v_j \bar{v}) dt.
\end{aligned} \tag{3.17}$$

Utilizing  $b^{jk} = b^{kj}$  once more, we find

$$\sum_{j,k,j',k'=1}^n b^{jk} b^{j'k'} \ell_j (v_{j'} \bar{v}_{kk'} + \bar{v}_{j'} v_{kk'}) = \sum_{j,k,j',k'=1}^n b^{jk} b^{j'k'} \ell_j (v_{j'k} \bar{v}_{k'} + \bar{v}_{j'k} v_{k'}).$$

Hence, we obtain that

$$\begin{aligned}
& 2 \sum_{j,k,j',k'=1}^n b^{jk} b^{j'k'} \ell_j (v_{j'} \bar{v}_{kk'} + \bar{v}_{j'} v_{kk'}) dt \\
& = \sum_{j,k,j',k'=1}^n b^{jk} b^{j'k'} \ell_j (v_{j'} \bar{v}_{kk'} + \bar{v}_{j'} v_{kk'}) dt + \sum_{j,k,j',k'=1}^n b^{jk} b^{j'k'} \ell_j (v_{j'k} \bar{v}_{k'} + \bar{v}_{j'k} v_{k'}) dt \\
& = \sum_{j,k,j',k'=1}^n b^{jk} b^{j'k'} \ell_j (v_{j'} \bar{v}_{k'} + \bar{v}_{j'} v_{k'})_k dt \\
& = \sum_{j,k,j',k'=1}^n \left[ b^{jk} b^{j'k'} \ell_j (v_{j'} \bar{v}_{k'} + \bar{v}_{j'} v_{k'}) \right]_k dt - \sum_{j,k,j',k'=1}^n (b^{jk} b^{j'k'} \ell_j)_k (v_{j'} \bar{v}_{k'} + \bar{v}_{j'} v_{k'}) dt.
\end{aligned} \tag{3.18}$$

By the equality (3.18), we get that

$$\begin{aligned}
& I_1^2 \overline{I_2^2} + I_2^2 \overline{I_1^2} \\
&= -2 \sum_{j,k=1}^n b^{jk} \ell_j v_k \overline{\sum_{j,k=1}^n (b^{jk} v_j)_k} dt - 2 \sum_{j,k=1}^n (b^{jk} v_j)_k \overline{\sum_{j,k=1}^n b^{jk} \ell_j v_k} dt \\
&= -2 \sum_{j,k,j',k'=1}^n \left[ b^{jk} b^{j'k'} \ell_j (v_{j'} \overline{v_k} + \overline{v_{j'}} v_k) \right]_{k'} dt + 2 \sum_{j,k,j',k'=1}^n b^{j'k'} (b^{jk} \ell_j)_{k'} (v_{j'} \overline{v_k} + \overline{v_{j'}} v_k) dt \\
&\quad + 2 \sum_{j,k,j',k'=1}^n b^{jk} b^{j'k'} \ell_j (v_{j'} \overline{v_{kk'}} + \overline{v_{j'}} v_{kk'}) dt \\
&= -2 \sum_{j,k,j',k'=1}^n \left[ b^{jk} b^{j'k'} \ell_j (v_{j'} \overline{v_k} + \overline{v_{j'}} v_k) \right]_{k'} dt + 2 \sum_{j,k,j',k'=1}^n b^{j'k'} (b^{jk} \ell_j)_{k'} (v_{j'} \overline{v_k} + \overline{v_{j'}} v_k) dt \\
&\quad + \sum_{j,k,j',k'=1}^n \left[ b^{jk} b^{j'k'} \ell_j (v_{j'} \overline{v_{k'}} + \overline{v_{j'}} v_{k'}) \right]_k dt - \sum_{j,k,j',k'=1}^n (b^{jk} b^{j'k'} \ell_j)_k (v_{j'} \overline{v_{k'}} + \overline{v_{j'}} v_{k'}) dt \\
&= -2 \sum_{j,k,j',k'=1}^n \left[ b^{jk'} b^{j'k} \ell_j (v_{j'} \overline{v_{k'}} + \overline{v_{j'}} v_{k'}) \right]_k dt + 2 \sum_{j,k,j',k'=1}^n b^{jk'} (b^{j'k} \ell_{j'})_{k'} (v_j \overline{v_k} + \overline{v_j} v_k) dt \\
&\quad + \sum_{j,k,j',k'=1}^n \left[ b^{jk} b^{j'k'} \ell_j (v_{j'} \overline{v_{k'}} + \overline{v_{j'}} v_{k'}) \right]_k dt - \sum_{j,k,j',k'=1}^n (b^{jk} b^{j'k'} \ell_{j'})_{k'} (v_j \overline{v_k} + \overline{v_j} v_k) dt.
\end{aligned} \tag{3.19}$$

Further, it holds that

$$\begin{aligned}
& I_1^3 \overline{I_2^2} + I_2^2 \overline{I_1^3} \\
&= \overline{\Psi v \sum_{j,k=1}^n (b^{jk} v_j)_k} dt + \sum_{j,k=1}^n (b^{jk} v_j)_k \overline{\Psi v} dt \\
&= \sum_{j,k=1}^n \left[ \Psi b^{jk} (v_j \overline{v} + \overline{v_j} v) \right]_k dt - \sum_{j,k=1}^n \Psi b^{jk} (v_j \overline{v_k} + \overline{v_j} v_k) dt \\
&\quad - \sum_{j,k=1}^n \Psi_k b^{jk} (v_j \overline{v} + \overline{v_j} v) dt \\
&= \sum_{j,k=1}^n \left[ \Psi b^{jk} (v_j \overline{v} + \overline{v_j} v) \right]_k dt - \sum_{j,k=1}^n \Psi b^{jk} (v_j \overline{v_k} + \overline{v_j} v_k) dt \\
&\quad - \sum_{j,k=1}^n \left[ b^{jk} \Psi_k |v|^2 \right]_j dt + \sum_{j,k=1}^n (b^{jk} \Psi_k)_j |v|^2 dt.
\end{aligned} \tag{3.20}$$

Finally, we have that

$$\begin{aligned}
 & I_1 \overline{I_2^3} + I_2^3 \overline{I_1} \\
 &= -i\beta \ell_t v \overline{A v} dt + A v \overline{(-i\beta \ell_t v)} dt \\
 &= -2 \sum_{j,k=1}^n (b^{jk} \ell_j A |v|^2)_k dt + 2 \left[ \sum_{j,k=1}^n (b^{jk} \ell_j A)_k + A \Psi \right] |v|^2 dt.
 \end{aligned} \tag{3.21}$$

**Step 3.** Combining (3.9)–(3.21), we conclude the desired formula (3.3).

**4. Carleman Estimate for Stochastic Schrödinger Equations.** This section is devoted to the proof of Theorem 1.3.

*Proof of Theorem 1.3:* The proof is divided into three steps.

**Step 1.** We choose  $\beta = 1$  and  $(b^{jk})_{1 \leq j,k \leq n}$  to be the identity matrix. Put

$$\delta^{jk} = \begin{cases} 1, & \text{if } j = k, \\ 0, & \text{if } j \neq k. \end{cases}$$

Applying Theorem 3.1 to the equation (1.1) with  $\theta$  given by (1.8),  $z$  replaced by  $y$  and  $v = \theta z$ . We obtain that

$$\begin{aligned}
 & \theta \mathcal{P} y \left( i\beta \ell_t \bar{v} - 2 \sum_{j,k=1}^n b^{jk} \ell_j \bar{v}_k + \Psi \bar{v} \right) + \theta \overline{\mathcal{P} y} \left( -i\beta \ell_t v - 2 \sum_{j,k=1}^n b^{jk} \ell_j v_k + \Psi v \right) \\
 &+ dM + \operatorname{div} V \\
 &= 2 \left| -i\beta \ell_t v - 2 \sum_{j,k=1}^n b^{jk} \ell_j v_k + \Psi v \right|^2 dt + \sum_{j,k=1}^n c^{jk} (v_k \bar{v}_j + \bar{v}_k v_j) dt + D|v|^2 dt \\
 &+ 2i \sum_{j=1}^n (\ell_{jt} + \ell_{tj}) (\bar{v}_j v - v_j \bar{v}) dt + i(\Psi + \Delta \ell) (\bar{v} dv - v d\bar{v}) \\
 &+ \ell_t dv d\bar{v} + i \sum_{j=1}^n \ell_j (d\bar{v}_j dv - dv_j d\bar{v}).
 \end{aligned} \tag{4.1}$$

Here

$$\begin{aligned}
 M &= \beta^2 \ell_t |v|^2 + i\beta \sum_{j,k=1}^n b^{jk} \ell_j (\bar{v}_k v - v_k \bar{v}) \\
 &= \ell_t |v|^2 + i \sum_{j=1}^n \ell_j (\bar{v}_j v - v_j \bar{v});
 \end{aligned} \tag{4.2}$$

$$\begin{aligned}
 A &= \sum_{j,k=1}^n b^{jk} \ell_j \ell_k - \sum_{j,k=1}^n (b^{jk} \ell_j)_k - \Psi \\
 &= \sum_{j=1}^n (\ell_j^2 - \ell_{jj}) - \Psi;
 \end{aligned} \tag{4.3}$$

$$\begin{aligned}
D &= (\beta^2 \ell_t)_t + \sum_{j,k=1}^n (b^{jk} \Psi_k)_j + 2 \left[ \sum_{j,k=1}^n (b^{jk} \ell_j A)_k + A \Psi \right] \\
&= \ell_{tt} + \sum_{j=1}^n \Psi_{jj} + 2 \sum_{j=1}^n (\ell_j A)_j + 2A \Psi;
\end{aligned} \tag{4.4}$$

$$\begin{aligned}
c^{jk} &= \sum_{j',k'=1}^n \left[ 2(b^{j'k} \ell_{j'})_{k'} b^{jk'} - (b^{jk} b^{j'k'} \ell_{j'})_{k'} \Psi \right] - b^{jk} \\
&= \left[ 2(b^{kk} \ell_k)_j b^{jj} - \sum_{j'=1}^n (b^{jk} b^{j'j'} \ell_{j'})_{j'} - b^{jk} \Psi \right] \\
&= 2\ell_{jk} - \delta^{jk} \Delta \ell - \delta^{jk} \Psi;
\end{aligned} \tag{4.5}$$

and

$$\begin{aligned}
V_k &= -i\beta \sum_{j=1}^n \left[ b^{jk} \ell_j (v d\bar{v} - \bar{v} dv) + b^{jk} \ell_t (v_j \bar{v} - \bar{v}_j v) dt \right] \\
&\quad - \Psi \sum_{j=1}^n b^{jk} (v_j \bar{v} + \bar{v}_j v) dt + \sum_{j=1}^n b^{jk} (2A \ell_j + \Psi_j) |v|^2 dt \\
&\quad + \sum_{j,j',k'=1}^n \left( 2b^{jk'} b^{j'k} - b^{jk} b^{j'k'} \right) \ell_j (v_{j'} \bar{v}_{k'} + \bar{v}_{j'} v_{k'}) dt \\
&= -i \left[ \ell_k (v d\bar{v} - \bar{v} dv) + \ell_t (v_j \bar{v} - \bar{v}_j v) dt \right] - \Psi (v_k \bar{v} + \bar{v}_k v) dt + (2A \ell_k + \Psi_k) |v|^2 dt \\
&\quad + 2 \sum_{j=1}^n \ell_j (\bar{v}_j v_k + v_j \bar{v}_k) dt - 2 \sum_{j'=1}^n \ell_k (v_j \bar{v}_{j'}) dt.
\end{aligned} \tag{4.6}$$

**Step 2.** In this step, we estimate the terms in the right-hand side of the equality (4.1) one by one.

First, from the definition of  $\ell$ ,  $\varphi$  (see (1.8)) and the choice of  $\psi$  (see (1.7)), we have that

$$\begin{aligned}
|\ell_t| &= \left| s \frac{2(2t-T)}{t^3(T-t)^3} (e^{4\lambda\psi} - e^{5\lambda|\psi|_{L^\infty(G)}}) \right| \\
&\leq \left| s \frac{2(2t-T)}{t^3(T-t)^3} e^{5\lambda|\psi|_{L^\infty(G)}} \right| \\
&\leq \left| s \frac{C}{t^3(T-t)^3} e^{5\lambda\psi} \right| \\
&\leq Cs\varphi^{1+\frac{1}{2}},
\end{aligned} \tag{4.7}$$

and that

$$\begin{aligned}
|\ell_{tt}| &= \left| s \frac{20t^2 - 20tT + 6T^2}{t^4(T-t)^4} (e^{4\lambda\psi} - e^{5\lambda|\psi|_{L^\infty(G)}}) \right| \\
&\leq \left| s \frac{C}{t^4(T-t)^4} e^{5\lambda|\psi|_{L^\infty(G)}} \right| \\
&\leq \left| s \frac{C}{t^4(T-t)^4} e^{8\lambda\psi} \right| \\
&\leq Cs\varphi^2 \leq Cs\varphi^3.
\end{aligned} \tag{4.8}$$

We choose below  $\Psi = -\Delta\ell$ , then we have that

$$A = \sum_{j=1}^m \ell_j^2 = \sum_{j=1}^m (4s\lambda\varphi\psi)^2 = 16s^2\lambda^2\varphi^2|\nabla\psi|^2. \tag{4.9}$$

Hence, we find

$$\begin{aligned}
D &= \ell_{tt} + \sum_{j=1}^n \Psi_{jj} + 2 \sum_{j=1}^n (\ell_j A)_j + 2A\Psi \\
&= \ell_{tt} + \Delta(\Delta\ell) + 2 \sum_{j=1}^n (4s\lambda\varphi\psi_j 16s^2\lambda^2\varphi^2|\nabla\psi|^2)_j - 32s^2\lambda^2\varphi^2|\nabla\psi|^2\Delta\ell \\
&= 384s^3\lambda^4\varphi^3|\nabla\psi|^4 - \lambda^4\varphi O(s) - s^3\varphi^3 O(\lambda^3) + \ell_{tt}.
\end{aligned} \tag{4.10}$$

Recalling that  $x_0 \in (\mathbb{R}^n \setminus \overline{G})$ , we know that

$$|\nabla\psi| > 0 \quad \text{in } \overline{G}.$$

From (4.10) and (4.8), we know that there exists a  $\lambda_0 > 0$  such that for all  $\lambda > \lambda_0$ , one can find a constant  $s_0 = s_0(\lambda_0)$  so that for any  $s > s_0$ , it holds that

$$D|v|^2 \geq s^3\lambda^4\varphi^3|\nabla\psi|^4|v|^2. \tag{4.11}$$

Since

$$\begin{aligned}
c^{jk} &= 2\ell_{jk} - \delta^{jk}\Delta\ell - \delta^{jk}\Psi \\
&= 32s\lambda^2\varphi\psi_j\psi_k + 16s\lambda\varphi\psi_{jk},
\end{aligned}$$

we see that

$$\begin{aligned}
&\sum_{j,k=1}^n c^{jk}(v_j\overline{v}_k + v_k\overline{v}_j) \\
&= 32s\lambda^2\varphi \sum_{j,k=1}^n \psi_j\psi_k(v_j\overline{v}_k + v_k\overline{v}_j) + 16s\lambda\varphi \sum_{j,k=1}^n \psi_{jk}(v_j\overline{v}_k + v_k\overline{v}_j) \\
&= 32s\lambda^2\varphi \left[ \sum_{j=1}^n (\psi_j v_j) \sum_{k=1}^n (\psi_k \overline{v}_k) + \sum_{k=1}^n (\psi_k v_k) \sum_{j=1}^n (\psi_j \overline{v}_j) \right] + 32s\lambda\varphi \sum_{j=1}^n (v_j \overline{v}_j + \overline{v}_j v_j) \\
&= 64s\lambda^2\varphi |\nabla\psi \cdot \nabla v|^2 + 64s\lambda\varphi |\nabla v|^2 \\
&\geq 64s\lambda\varphi |\nabla v|^2.
\end{aligned} \tag{4.12}$$

Now we estimate the other terms in the right-hand side of the equality (4.1). The first one satisfies that

$$\begin{aligned} 2i \sum_{j=1}^n (\ell_{jt} + \ell_{tj})(\bar{v}_j v - v_j \bar{v}) &= 4i \sum_{j=1}^n s\lambda\psi_j \ell_t (\bar{v}_j v - \bar{v} v_j) \\ &\leq 2s\varphi |\nabla v|^2 + 2s\lambda^2 \varphi^3 |\nabla \psi|^2 |v|^2. \end{aligned} \quad (4.13)$$

The second one reads

$$i(\Psi + \Delta \ell)(\bar{v} dv - v d\bar{v}) = 0. \quad (4.14)$$

For the estimate of the third and the fourth one, we need to take mean value and get that

$$\begin{aligned} \mathbb{E}(\ell_t dv d\bar{v}) &= \mathbb{E}[\ell_t(\theta \ell_t y dt + \theta dy)(\overline{\theta \ell_t y dt + \theta dy})] = \mathbb{E}(\ell_t \theta^2 dy d\bar{y}) \\ &\leq 2s\theta^2 \varphi^{\frac{3}{2}} \mathbb{E}(a_3^2 |y|^2 + g^2) dt. \end{aligned} \quad (4.15)$$

Here we utilize inequality (4.7).

Since

$$\begin{aligned} \mathbb{E}(d\bar{v}_j dv) &= \mathbb{E}[\overline{(\theta \ell_t v dt + \theta dy)}_j (\theta \ell_t v dt + \theta dy)] \\ &= \mathbb{E}[\overline{(\theta dy)}_j (\theta dy)] \\ &= \mathbb{E}[\overline{(s\lambda\varphi\psi_j \theta dy + \theta dy_j)} \theta dy] \\ &= s\lambda\varphi\psi_j \theta^2 \mathbb{E}d\bar{y} dy + \theta^2 \mathbb{E}d\bar{y}_j dy \\ &= s\lambda\varphi\psi_j \theta^2 \mathbb{E}|a_3 y + g|^2 dt + \theta^2 \mathbb{E}[\overline{(a_3 y + g)}_j (a_3 y + g)] dt \end{aligned}$$

and

$$\begin{aligned} &\theta^2 \mathbb{E}[\overline{(a_3 y + g)}_j (a_3 y + g)] dt \\ &= \theta^2 \mathbb{E}[(\overline{a_3 y})_j (a_3 y) + (\overline{a_3 y})_j g + (a_3 y) \bar{g}_j + g \bar{g}_j] dt \\ &= \theta^2 \mathbb{E}[(\overline{a_3 y})_j (a_3 y) + (\overline{a_3 y})_j g + g \bar{g}_j] dt + [\mathbb{E}\theta^2 (a_3 y) \bar{g}]_j \\ &\quad - s\lambda\varphi\psi_j \theta^2 \mathbb{E}(a_3 y \bar{g}) - \theta^2 \mathbb{E}[(a_3 y)_j \bar{g}], \end{aligned}$$

we get that

$$\begin{aligned} \mathbb{E}(d\bar{v}_j dv) &= s\lambda\varphi\psi_j \theta^2 \mathbb{E}|a_3 y + g|^2 dt + \theta^2 \mathbb{E}[(\overline{a_3 y})_j (a_3 y) + (\overline{a_3 y})_j g + g \bar{g}_j] dt \\ &\quad + \mathbb{E}(\theta^2 a_3 y \bar{g})_j - s\lambda\varphi\psi_j \theta^2 \mathbb{E}(a_3 y \bar{g}) - \theta^2 \mathbb{E}[(a_3 y)_j \bar{g}]. \end{aligned}$$

Similarly, we can get that

$$\begin{aligned} \mathbb{E}(dv_j d\bar{v}) &= s\lambda\varphi\psi_j \theta^2 \mathbb{E}|a_3 y + g|^2 dt + \theta^2 \mathbb{E}[(\overline{a_3 y})(a_3 y)_j + (a_3 y)_j \bar{g} + g_j \bar{g}] dt \\ &\quad + \mathbb{E}(\theta^2 \overline{a_3 y} g)_j - s\lambda\varphi\psi_j \theta^2 \mathbb{E}(\overline{a_3 y} g) - \theta^2 \mathbb{E}[(\overline{a_3 y})_j g]. \end{aligned}$$



Therefore, fourth one enjoys that

$$\begin{aligned}
 & i\mathbb{E} \sum_{j=1}^n \ell_j (d\bar{v}_j dv - dv_j d\bar{v}) \\
 &= s\lambda\varphi \sum_{j=1}^n \psi_j \left[ \mathbb{E}(d\bar{v}_j dv) - \mathbb{E}(dv_j d\bar{v}) \right] \\
 &= s\lambda\varphi\psi \sum_{j=1}^n \psi_j \theta^2 \mathbb{E} \left\{ [(\overline{a_3 y})_j (a_3 y) + (\overline{a_3 y})_j g + g\bar{g}_j - s\lambda\varphi\psi_j a_3 y \bar{g} - (a_3 y)_j \bar{g}] \right. \\
 &\quad \left. - [(\overline{a_3 y})(a_3 y)_j + (a_3 y)_j \bar{g} + g_j \bar{g} - s\lambda\varphi\psi_j (\overline{a_3 y} g) - [(\overline{a_3 y})_j g]] \right\} dt \\
 &\quad + s\lambda\varphi\psi \sum_{j=1}^n \psi_j \mathbb{E} (\theta^2 a_3 y \bar{g} - \theta^2 \overline{a_3 y} g)_j.
 \end{aligned} \tag{4.16}$$

**Step 3.** Integrating the equality (4.1) in  $Q$ , taking mean value in both sides, and noting (4.9)–(4.16), we obtain that

$$\begin{aligned}
 & \mathbb{E} \int_Q \left( s^3 \lambda^4 \varphi^3 |v|^2 + s\lambda^2 \varphi |\nabla v|^2 \right) dx dt + 2\mathbb{E} \int_Q \left| -i\beta \ell_t v - 2 \sum_{j,k=1}^n b^{jk} \ell_j v_k + \Psi v \right|^2 dx dt \\
 & \leq \mathbb{E} \int_Q \left\{ \theta \mathcal{P}y \left( i\beta \ell_t \bar{v} - 2 \sum_{j,k=1}^n b^{jk} \ell_j \bar{v}_k + \Psi \bar{v} \right) + \theta \overline{\mathcal{P}y} \left( -i\beta \ell_t v - 2 \sum_{j,k=1}^n b^{jk} \ell_j v_k + \Psi v \right) \right\} dx \\
 & \quad + C\mathbb{E} \int_Q \theta^2 \left[ s^2 \lambda^2 \varphi^2 (a_3^2 |y|^2 + g^2) + a_3^2 |\nabla y|^2 + |\nabla a_3|^2 y^2 + |\nabla g|^2 \right] dx dt \\
 & \quad + \mathbb{E} \int_Q dM dx + \mathbb{E} \int_Q \operatorname{div} V dx.
 \end{aligned} \tag{4.17}$$

Now we analyze the terms in the right-hand side of the inequality (4.17) one by one.

The first term satisfies that

$$\begin{aligned}
 & \mathbb{E} \int_Q \left\{ \theta \mathcal{P}y \left( i\beta \ell_t \bar{v} - 2 \sum_{j,k=1}^n b^{jk} \ell_j \bar{v}_k + \Psi \bar{v} \right) \right. \\
 & \quad \left. + \theta \overline{\mathcal{P}y} \left( -i\beta \ell_t v - 2 \sum_{j,k=1}^n b^{jk} \ell_j v_k + \Psi v \right) \right\} dx \\
 &= \mathbb{E} \int_Q \left\{ \theta (a_1 \cdot \nabla y + a_2 y + f) \left( i\beta \ell_t \bar{v} - 2 \sum_{j,k=1}^n b^{jk} \ell_j \bar{v}_k + \Psi \bar{v} \right) \right. \\
 & \quad \left. + \theta (a_1 \cdot \nabla \bar{y} + \overline{a_2 y} + \bar{f}) \left( -i\beta \ell_t v - 2 \sum_{j,k=1}^n b^{jk} \ell_j v_k + \Psi v \right) \right\} dx dt \\
 & \leq 2\mathbb{E} \int_Q \left\{ \theta^2 |a_1 \cdot \nabla y + a_2 y + f|^2 + \left| -i\beta \ell_t v - 2 \sum_{j,k=1}^n b^{jk} \ell_j v_k + \Psi v \right|^2 \right\} dx dt.
 \end{aligned}$$

From the definition of  $\theta$ , we know that  $v(0) = v(T) = 0$ . Hence, it holds that

$$\int_Q dM dx = 0. \quad (4.18)$$

By means of Stokes' Theorem, we have that

$$\begin{aligned} \mathbb{E} \int_Q \operatorname{div} V dx &= \mathbb{E} \int_{\Sigma} 2 \sum_{k=1}^n \sum_{j=1}^n \left[ \ell_j (\bar{v}_j v_k + v_j \bar{v}_k) \nu^k - \ell_k \nu_k v_j \bar{v}_j \right] d\Sigma \\ &= \mathbb{E} \int_{\Sigma} \left( 4 \sum_{j=1}^n \ell_j \nu_j \left| \frac{\partial v}{\partial \nu} \right|^2 - 2 \sum_{k=1}^n \ell_k \nu_k \left| \frac{\partial v}{\partial \nu} \right|^2 \right) d\Sigma \\ &= \mathbb{E} \int_{\Sigma} 2 \sum_{k=1}^n \ell_k \nu_k \left| \frac{\partial v}{\partial \nu} \right|^2 d\Sigma \\ &\leq C \mathbb{E} \int_0^T \int_{\Gamma_0} \theta^2 s \lambda \varphi \left| \frac{\partial y}{\partial \nu} \right|^2 d\Gamma dt. \end{aligned} \quad (4.19)$$

By (4.17)–(4.19), we have that

$$\begin{aligned} &\mathbb{E} \int_Q \left( s^3 \lambda^4 \varphi^3 |v|^2 + s \lambda \varphi |\nabla v|^2 \right) dx dt \\ &\leq C \mathbb{E} \int_Q \theta^2 |a_1 \cdot \nabla y + a_2 y + f|^2 dx dt + C \mathbb{E} \int_0^T \int_{\Gamma_0} \theta^2 s \lambda \varphi \left| \frac{\partial y}{\partial \nu} \right|^2 d\Gamma dt \\ &\quad + C \mathbb{E} \int_Q \theta^2 \left[ s^2 \lambda^2 \varphi^2 (a_3^2 |y|^2 + g^2) + a_3^2 |\nabla y|^2 + |\nabla a_3|^2 y^2 + |\nabla g|^2 \right] dx dt. \end{aligned} \quad (4.20)$$

Noting that  $y_i = \theta^{-1}(v_i - \ell_i v) = \theta^{-1}(v_i - s \lambda \varphi \psi_i v)$ , we get

$$\theta^2 (|\nabla y|^2 + s^2 \lambda^2 \varphi^2 |y|^2) \leq C (|\nabla v|^2 + s^2 \lambda^2 \varphi^2 |v|^2). \quad (4.21)$$

Therefore, it follows from (4.20) that

$$\begin{aligned} &\mathbb{E} \int_Q \left( s^3 \lambda^4 \varphi^3 |y|^2 + s \lambda \varphi |\nabla y|^2 \right) dx dt \\ &\leq C \mathbb{E} \int_Q \theta^2 (|a_1|^2 |\nabla y|^2 + a_2^2 |y|^2 + |f|^2) dx dt + C \mathbb{E} \int_0^T \int_{\Gamma_0} \theta^2 s \lambda \varphi \left| \frac{\partial y}{\partial \nu} \right|^2 d\Gamma dt \\ &\quad + C \mathbb{E} \int_Q \theta^2 \left[ s^2 \lambda^2 \varphi^2 (a_3^2 |y|^2 + g^2) + a_3^2 |\nabla y|^2 + |\nabla a_3|^2 y^2 + |\nabla g|^2 \right] dx dt. \end{aligned} \quad (4.22)$$

Taking  $\lambda_1 = \lambda_0$  and  $s_1 = \max(s_0, Cr_1)$ , and utilizing the inequality (4.22), we conclude the desired inequality (1.9).

On the other hand, if  $g \in L^2_{\mathcal{F}}(0, T; H^1(G; \mathbb{R}))$ , then  $g\bar{g}_j - g_j\bar{g} = 0$  for  $j = 1, \dots, n$ .

Thus, from (4.9)–(4.16), we get

$$\begin{aligned}
& \mathbb{E} \int_Q \left( s^3 \lambda^4 \varphi^3 |v|^2 + s \lambda^2 \varphi |\nabla v|^2 \right) dx dt + 2 \mathbb{E} \int_Q \left| -i \beta \ell_t v - 2 \sum_{j,k=1}^n b^{jk} \ell_j v_k + \Psi v \right|^2 dx dt \\
& \leq \mathbb{E} \int_Q \left\{ \theta \mathcal{P} y \left( i \beta \ell_t \bar{v} - 2 \sum_{j,k=1}^n b^{jk} \ell_j \bar{v}_k + \Psi \bar{v} \right) + \theta \overline{\mathcal{P} y} \left( -i \beta \ell_t v - 2 \sum_{j,k=1}^n b^{jk} \ell_j v_k + \Psi v \right) \right\} dx \\
& \quad + C \mathbb{E} \int_Q \theta^2 \left[ s^2 \lambda^2 \varphi^2 (a_3^2 |y|^2 + g^2) + a_3^2 |\nabla y|^2 + |\nabla a_3|^2 y^2 \right] dx dt + \mathbb{E} \int_Q dM dx \\
& \quad + \mathbb{E} \int_Q \operatorname{div} V dx.
\end{aligned} \tag{4.23}$$

Then, by a similar argument, we find that

$$\begin{aligned}
& \mathbb{E} \int_Q \left( s^3 \lambda^4 \varphi^3 |y|^2 + s \lambda \varphi |\nabla y|^2 \right) dx dt \\
& \leq C \mathbb{E} \int_Q \theta^2 \left( |a_1|^2 |\nabla y|^2 + a_2^2 |y|^2 + |f|^2 \right) dx dt + C \mathbb{E} \int_0^T \int_{\Gamma_0} \theta^2 s \lambda \varphi \left| \frac{\partial y}{\partial \nu} \right|^2 d\Gamma dt \\
& \quad + C \mathbb{E} \int_Q \theta^2 \left[ s^2 \lambda^2 \varphi^2 (a_3^2 |y|^2 + g^2) + a_3^2 |\nabla y|^2 + |\nabla a_3|^2 y^2 \right] dx dt.
\end{aligned} \tag{4.24}$$

Now taking  $\lambda_1 = \lambda_0$  and  $s_1 = \max(s_0, Cr_1)$ , and using the inequality (4.24), we obtain the desired inequality (1.10).

**5. Proof of Theorem 1.2.** In this section, we prove Theorems 1.2, by means of Theorem 1.3.

*Proof of Theorem 1.2:* By means of the definition of  $\ell$  and  $\theta$  (see (1.8)), it holds that

$$\begin{aligned}
& \mathbb{E} \int_Q \theta^2 \left( \varphi^3 |y|^2 + \varphi |\nabla y|^2 \right) dx dt \\
& \geq \min_{x \in \overline{G}} \left( \varphi \left( \frac{T}{2}, x \right) \theta^2 \left( \frac{T}{4}, x \right) \right) \mathbb{E} \int_{\frac{T}{4}}^{\frac{3T}{4}} \int_G (|y|^2 + |\nabla y|^2) dx dt,
\end{aligned} \tag{5.1}$$

$$\begin{aligned}
& \mathbb{E} \int_Q \theta^2 (|f|^2 + \varphi^2 |g|^2 + |\nabla g|^2) dx dt \\
& \leq \max_{(x,t) \in \overline{Q}} (\varphi^2(t,x) \theta^2(t,x)) \mathbb{E} \int_Q (|f|^2 + |g|^2 + |\nabla g|^2) dx dt
\end{aligned} \tag{5.2}$$

and that

$$\mathbb{E} \int_0^T \int_{\Gamma_0} \theta^2 \varphi \left| \frac{\partial y}{\partial \nu} \right|^2 d\Gamma dt \leq \max_{(x,t) \in \overline{Q}} (\varphi(t,x) \theta^2(t,x)) \mathbb{E} \int_0^T \int_{\Gamma_0} \left| \frac{\partial y}{\partial \nu} \right|^2 d\Gamma dt. \tag{5.3}$$

From (1.9) and (5.1)–(5.3), we deduce that

$$\begin{aligned}
& \mathbb{E} \int_{\frac{T}{4}}^{\frac{3T}{4}} \int_G (|y|^2 + |\nabla y|^2) dx dt \\
& \leq Cr_1 \frac{\max_{(x,t) \in \overline{Q}} \left( \varphi^2(t, x) \theta^2(t, x) \right)}{\min_{x \in \overline{G}} \left( \varphi(\frac{T}{2}, x) \theta^2(\frac{T}{4}, x) \right)} \\
& \quad \times \left\{ \mathbb{E} \int_Q (|f|^2 + |g|^2 + |\nabla g|^2) dx dt + \mathbb{E} \int_0^T \int_{\Gamma_0} \left| \frac{\partial y}{\partial \nu} \right|^2 d\Gamma dt \right\} \\
& \leq e^{Cr_1} \left\{ \mathbb{E} \int_Q (|f|^2 + |g|^2 + |\nabla g|^2) dx dt + \mathbb{E} \int_0^T \int_{\Gamma_0} \left| \frac{\partial y}{\partial \nu} \right|^2 d\Gamma dt \right\}.
\end{aligned} \tag{5.4}$$

Utilizing (5.4) and (2.1), we obtain that

$$\begin{aligned}
& \mathbb{E} \int_G (|y_0|^2 + |\nabla y_0|^2) dx \\
& \leq e^{Cr_1} \left\{ \mathbb{E} \int_Q (|f|^2 + |\nabla f|^2 + |g|^2 + |\nabla g|^2) dx dt + \mathbb{E} \int_0^T \int_{\Gamma_0} \left| \frac{\partial y}{\partial \nu} \right|^2 d\Gamma dt \right\},
\end{aligned} \tag{5.5}$$

which concludes Theorem 1.2 immediately.  $\square$

**6. Two applications.** This section is addressed to applications of the observability/Carleman estimates shown in Theorems 1.2–1.3.

We first study a state observation problem for semilinear stochastic Schrödinger equations. Let us consider the following equation:

$$\begin{cases} idz + \Delta z dt = [a_1 \cdot \nabla z + a_2 z + F_1(|z|)] dt + [a_3 z + F_2(|z|)] dB(t) & \text{in } Q, \\ z = 0 & \text{on } \Sigma, \\ z(0) = z_0, & \text{in } G. \end{cases} \tag{6.1}$$

Here  $a_i$  ( $i = 1, 2, 3$ ) are given as in (1.3),  $F_1(\cdot) \in C^1(\mathbb{R}; \mathbb{C})$  with  $F(0) = 0$  and  $F_2(\cdot) \in C^1(\mathbb{R}; \mathbb{R})$  are two known nonlinear global Lipschitz continuous functions with Lipschitzian constant  $L$ , while the initial datum  $z_0 \in L^2(\Omega, \mathcal{F}_0, P; H_0^1(G))$  is unknown. The solution to the equation (6.1) is understood similar to Definition 1.1.

**REMARK 6.1.** *From the choice of  $F_1$  and  $F_2$ , one can easily show that the equation (6.1) admits a unique solution  $z \in H_T$  by the standard fixed point argument. We omit the proof here.*

The state observation problem associated to the equation (6.1) is as follows.

- **Identifiability.** Is the solution  $z \in H_T$  (to (6.1)) determined uniquely by the observation  $\frac{\partial z}{\partial \nu} \Big|_{(0,T) \times \Gamma_0}$ ?
- **Stability.** Assume that two solutions  $z$  and  $\hat{z}$  (to (6.1)) are given, and let  $\frac{\partial z}{\partial \nu} \Big|_{(0,T) \times \Gamma_0}$  and  $\frac{\partial \hat{z}}{\partial \nu} \Big|_{(0,T) \times \Gamma_0}$  be the corresponding observations. Can we find a positive constant  $C$  such that

$$\|z - \hat{z}\| \leq C \left\| \frac{\partial z}{\partial \nu} - \frac{\partial \hat{z}}{\partial \nu} \right\|,$$

with appropriate norms in both sides?

- **Reconstruction.** Is it possible to reconstruct  $z \in H_T$  to (6.1), in some sense, from the observation  $\frac{\partial z}{\partial \nu}|_{(0,T) \times \Gamma_0}$ ?

The state observation problem for systems governed by deterministic partial differential equations is studied extensively (See [12, 17, 25] and the rich references therein). However, the stochastic case attracts very little attention. To our best knowledge, [30] is the only published paper addressing this topic. In that paper, the author studied the state observation problem for semilinear stochastic wave equations. By means of Theorem 1.2, we can give positive answers to the above first and second questions.

We claim that  $\frac{\partial z}{\partial \nu}|_{(0,T) \times \Gamma_0} \in L^2_{\mathcal{F}}(0, T; L^2(\Gamma_0))$  (and therefore, the observation makes sense). Indeed, from the choice of  $F_1$ , it follows that

$$\begin{aligned} \mathbb{E} \int_0^T \int_G |\nabla(F_1(|z|))|^2 dx dt &= \mathbb{E} \int_0^T \int_G |F'_1(|z|) \nabla |z||^2 dx dt \leq L \mathbb{E} \int_0^T \int_G |\nabla |z||^2 dx dt \\ &\leq L \mathbb{E} \int_0^T \int_G |\nabla z|^2 dx dt, \end{aligned}$$

and

$$F(|z(t, \cdot)|) = 0 \quad \text{on } \Gamma \text{ for a.e. } t \in [0, T].$$

Hence,

$$F_1(|z|) \in L^2_{\mathcal{F}}(0, T; H_0^1(G)) \text{ for any } z \in H_T.$$

Similarly,

$$F_2(|z|) \in L^2_{\mathcal{F}}(0, T; H^1(G)) \text{ for any } z \in H_T.$$

Consequently, by Proposition 2.2, we find that  $\frac{\partial z}{\partial \nu}|_{(0,T) \times \Gamma_0} \in L^2_{\mathcal{F}}(0, T; L^2(\Gamma_0))$ .

Now, we define a nonlinear map as follows:

$$\begin{cases} \mathcal{M} : L^2(\Omega, \mathcal{F}_0, P; H_0^1(G)) \rightarrow L^2_{\mathcal{F}}(0, T; L^2(\Gamma_0)), \\ \mathcal{M}(z_0) = \frac{\partial z}{\partial \nu}|_{(0,T) \times \Gamma_0}, \end{cases}$$

where  $z$  solves the equation (6.1). We have the following result.

**THEOREM 6.1.** *There exists a constant  $\tilde{C} = \tilde{C}(L, T, G) > 0$  such that for any  $z_0, \hat{z}_0 \in L^2(\Omega, \mathcal{F}_0, P; H_0^1(G))$ , it holds that*

$$|z_0 - \hat{z}_0|_{L^2(\Omega, \mathcal{F}_0, P; L^2(G))} \leq \tilde{C} |\mathcal{M}(z_0) - \mathcal{M}(\hat{z}_0)|_{L^2_{\mathcal{F}}(0, T; L^2(\Gamma_0))}, \quad (6.2)$$

where  $\hat{z} = \hat{z}(\cdot; \hat{z}_0) \in H_T$  is the solution to (6.1) with  $z_0$  replaced by  $\hat{z}_0$ .

**REMARK 6.2.** *From the well-posedness of the equation (6.1), Theorem 6.1 indicates that the state  $z(t)$  of (6.1) (for  $t \in [0, T]$ ) can be uniquely determined from the observed boundary data  $\frac{\partial z}{\partial \nu}|_{(0,T) \times \Gamma_0}$ ,  $P$ -a.s., and continuously depends on it. Therefore, we answer the first and second questions for state observation problem for the system (6.1) positively.*

*Proof of Theorem 6.1:* Set

$$y = z - \hat{z}.$$

Then, it is easy to see that  $y$  satisfies

$$\begin{cases} idy + \Delta y dt = [a_1 \cdot \nabla y + a_2 y + F_1(|z|) - F_1(|\hat{z}|)] dt \\ \quad + [a_3 y + F_2(|z|) - F_2(|\hat{z}|)] dB(t) & \text{in } Q, \\ y = 0 & \text{on } \Sigma, \\ y(0) = z_0 - \hat{z}_0 & \text{in } G. \end{cases}$$

Also, it is clear that

$$F_1(|z|) - F_1(|\hat{z}|) \in L^2_{\mathcal{F}}(0, T; H^1_0(G))$$

and

$$F_2(|z|) - F_2(|\hat{z}|) \in L^2_{\mathcal{F}}(0, T; H^1(G)).$$

Hence, we know that  $y$  solves the equation (1.1) with

$$\begin{cases} f = F_1(|z|) - F_1(|\hat{z}|), \\ g = F_2(|z|) - F_2(|\hat{z}|). \end{cases}$$

By means of the inequality (1.10) in Theorem 1.3, there exist an  $s_1 > 0$  and a  $\lambda_1 > 0$  so that for all  $s \geq s_1$  and  $\lambda \geq \lambda_1$ , it holds that

$$\begin{aligned} & \mathbb{E} \int_Q \theta^2 (s^3 \lambda^4 \varphi^3 |y|^2 + s \lambda \varphi |\nabla y|^2) dx dt \\ & \leq C \left\{ \mathbb{E} \int_Q \theta^2 (|f|^2 + s^2 \lambda^2 \varphi^2 |g|^2) dx dt + \mathbb{E} \int_0^T \int_{\Gamma_0} \theta^2 s \lambda \varphi \left| \frac{\partial y}{\partial \nu} \right|^2 d\Gamma dt \right\}. \end{aligned}$$

By the choice of  $f$ , we see that

$$\begin{aligned} \mathbb{E} \int_Q \theta^2 |f|^2 dx dt & \leq \mathbb{E} \int_Q \theta^2 |F_1(|z|) - F_1(|\hat{z}|)|^2 dx dt \leq L \mathbb{E} \int_Q \theta^2 (|z| - |\hat{z}|)^2 dx dt \\ & \leq L \mathbb{E} \int_Q \theta^2 |z - \hat{z}|^2 dx dt \leq L \mathbb{E} \int_Q \theta^2 |y|^2 dx dt. \end{aligned}$$

Similarly,

$$s^2 \lambda^2 \mathbb{E} \int_Q \theta^2 \varphi^2 |g|^2 dx dt \leq L \mathbb{E} \int_Q \theta^2 \varphi^2 |y|^2 dx dt.$$

Hence, we obtain that

$$\begin{aligned} & \mathbb{E} \int_Q \theta^2 (s^3 \lambda^4 \varphi^3 |y|^2 + s \lambda \varphi |\nabla y|^2) dx dt \\ & \leq C \left\{ L \mathbb{E} \int_Q \theta^2 (|y|^2 + s^2 \lambda^2 \varphi^2 |y|^2) dx dt + \mathbb{E} \int_0^T \int_{\Gamma_0} \theta^2 s \lambda \varphi \left| \frac{\partial y}{\partial \nu} \right|^2 d\Gamma dt \right\}. \end{aligned}$$

Thus, there is a  $\lambda_2 \geq \max\{\lambda_1, CL\}$  such that for all  $s \geq s_1$  and  $\lambda \geq \lambda_2$ , it holds that

$$\mathbb{E} \int_Q \theta^2 \left( s^3 \lambda^4 \varphi^3 |y|^2 + s \lambda \varphi |\nabla y|^2 \right) dx dt \leq C \mathbb{E} \int_0^T \int_{\Gamma_0} \theta^2 s \lambda \varphi \left| \frac{\partial y}{\partial \nu} \right|^2 d\Gamma dt. \quad (6.3)$$

Further, similar to the proof of the inequality (2.2), we can obtain that for any  $0 \leq t \leq s \leq T$ , it holds

$$\begin{aligned} \mathbb{E}|y(t)|_{L^2(G)}^2 - \mathbb{E}|y(s)|_{L^2(G)}^2 &\leq 2\mathbb{E} \int_t^s \int_G \left[ |f|^2 + |g|^2 \right] dx d\sigma \\ &\leq CL \mathbb{E} \int_t^s \int_G |y|^2 dx d\sigma. \end{aligned} \quad (6.4)$$

Then, by Gronwall's inequality, we find that

$$\mathbb{E}|y(t)|_{L^2(G)}^2 \leq e^{CL} \mathbb{E}|y(s)|_{L^2(G)}^2, \text{ for any } 0 \leq t \leq s \leq T. \quad (6.5)$$

Combining (6.3) and (6.5), similar as the derivation of the inequality (5.5), we obtain the inequality (6.2).  $\square$

Now we consider the unique continuation property for the equation (1.1). There are numerous works on the unique continuation property for deterministic partial differential equations. The study in this respect began at the very beginning of the 20th century; while a climax appeared in the last 1950-70's. The most powerful tools at that period is the local Carleman estimate (See [11] for example). Nevertheless, most of the existing works are devoted to the local unique continuation property at that time. In the recent 20 years, motivated by Control/Inverse Problems of partial differential equations, the study of the global unique continuation is very active (See [4, 26, 31] and the rich references therein). Compared with the fruitful works on the unique continuation property in the deterministic settings, there exist few results for stochastic partial differential equations. As far as we know, [28, 29] are the only two published articles addressed to this topic, and there is no result on the global unique continuation property for stochastic Schrödinger equations in the previous literature.

We remark that the powerful approach based on local Carleman estimate in the deterministic settings is very hard to apply to the stochastic counterpart. Indeed, the usual approach to employ local Carleman estimate for the unique continuation needs to localize the problem. Unfortunately, one cannot simply localize the problem as usual in the stochastic situation, since the usual localization technique may change the adaptedness of solutions, which is a key feature in the stochastic setting. In this paper, as a consequence of Theorem 1.2 (which is based on the global Carleman estimate established in Theorem 1.3), we obtain the following unique continuation property for solutions to the equation (1.1).

**THEOREM 6.2.** *For any  $\varepsilon > 0$ , let*

$$O_\varepsilon([0, T] \times \Gamma_0) \triangleq \left\{ (x, t) \in Q : \text{dist}((x, t), [0, T] \times \Gamma_0) \leq \varepsilon \right\}.$$

*Let  $f = g = 0$ ,  $P$ -a.s. For any  $y$  which solves the equation (1.1), if*

$$y = 0 \quad \text{in } O_\varepsilon([0, T] \times \Gamma_0), \quad P\text{-a.s.}, \quad (6.6)$$

*then  $y = 0$  in  $Q$ ,  $P$ -a.s.*

*Proof:* By (6.6), we see that  $\frac{\partial y}{\partial \nu} = 0$  on  $(0, T) \times \Gamma_0$ ,  $P$ -a.s. Hence, by means of Theorem 1.2, we find that  $y(0) = 0$  in  $L^2(\Omega, \mathcal{F}_0, P; H_0^1(G))$ . Consequently, we conclude that  $y = 0$  in  $Q$ ,  $P$ -a.s.  $\square$

**7. Further comments and open problems.** The subject of this paper is full of open problems. Some of them seem to be particularly relevant and could need important new ideas and further developments:

- **Observability estimate for backward stochastic Schrödinger equations**

Compared with Theorem 1.2, it is more interesting and difficult to establish the boundary observability estimate for backward stochastic Schrödinger equations. More precisely, let us consider the following backward stochastic Schrödinger equation:

$$\begin{cases} idu + \Delta u dt = (a_1 \cdot \nabla u + a_2 u + f)dt + (a_3 u + U + g)dB(t) & \text{in } Q, \\ u = 0 & \text{on } \Sigma, \\ u(T) = u_T & \text{in } G. \end{cases} \quad (7.1)$$

Here the final state  $u_T \in L^2(\Omega, \mathcal{F}_T, P; H_0^1(G))$  and  $\{\mathcal{F}_t\}_{t \geq 0}$  is the natural filtration generated by  $\{B(t)\}_{t \geq 0}$ . We expect the following result:

*Under the assumptions (1.2)–(1.5), any solution of the equation (7.1) satisfies that*

$$\begin{aligned} & |u_T|_{L^2(\Omega, \mathcal{F}_T, P; H_0^1(G))} \\ & \leq e^{Cr_1} \left( \left| \frac{\partial u}{\partial \nu} \right|_{L_{\mathcal{F}}^2(0, T; L^2(\Gamma_0))} + |f|_{L_{\mathcal{F}}^2(0, T; H_0^1(G))} + |g|_{L_{\mathcal{F}}^2(0, T; H^1(G))} \right), \end{aligned} \quad (7.2)$$

or at least,

$$\begin{aligned} & |u(0)|_{L^2(\Omega, \mathcal{F}_0, P; H_0^1(G))} \\ & \leq e^{Cr_1} \left( \left| \frac{\partial u}{\partial \nu} \right|_{L_{\mathcal{F}}^2(0, T; L^2(\Gamma_0))} + |f|_{L_{\mathcal{F}}^2(0, T; H_0^1(G))} + |g|_{L_{\mathcal{F}}^2(0, T; H^1(G))} \right). \end{aligned} \quad (7.3)$$

Unfortunately, following the method in this paper, one could obtain only an inequality as follows:

$$\begin{aligned} & |u_T|_{L^2(\Omega, \mathcal{F}_T, P; H_0^1(G))} \\ & \leq e^{Cr_1} \left( \left| \frac{\partial u}{\partial \nu} \right|_{L_{\mathcal{F}}^2(0, T; L^2(\Gamma_0))} + |U|_{L_{\mathcal{F}}^2(0, T; H^1(G))} + |f|_{L_{\mathcal{F}}^2(0, T; H_0^1(G))} \right. \\ & \quad \left. + |g|_{L_{\mathcal{F}}^2(0, T; H^1(G))} \right). \end{aligned} \quad (7.4)$$

It seems to us that getting rid of the undesired term  $|U|_{L_{\mathcal{F}}^2(0, T; H^1(G))}$  in the inequality (7.4) is a very challenging task.

- **Construction of the solution  $z$  from the observation**

In this paper, we only answer the first and the second questions in the state observation problem. The third one is still open. Since the equation (6.1) is time irreversible, some efficient approaches (See [17] for example), which work well for time reversible systems, become invalid. On the other hand, we may consider the following minimization problem:

Find a  $\bar{z}_0 \in L^2(\Omega, \mathcal{F}_0, P; H_0^1(G))$  such that

$$\left| \frac{\partial \bar{z}}{\partial \nu} - h \right|_{L_{\mathcal{F}}^2(0, T; L^2(\Gamma_0))} = \min_{z_0 \in L^2(\Omega, \mathcal{F}_0, P; H_0^1(G))} \left| \frac{\partial z}{\partial \nu} - h \right|_{L_{\mathcal{F}}^2(0, T; L^2(\Gamma_0))},$$



where  $h \in L^2_{\mathcal{F}}(0, T; L^2(\Gamma_0))$  is the observation and  $z$  (resp.  $\bar{z}$ ) is the solution to the equation (6.1) with initial datum  $z_0$  (resp.  $\bar{z}_0$ ).

It seems that one may utilize the method from optimization theory to study the construction of  $z_0$ . Because of the stochastic nature, this is an interesting but difficult problem and the detailed analysis is beyond the scope of this paper.

• **Unique continuation property with less restrictive conditions**

In this paper, we show that, under the condition (6.6),  $y = 0$  in  $Q$ ,  $P$ -a.s. Compared to the classical unique continuation result for deterministic Schrödinger equations with time independent coefficients (see [6, 16] for example), the condition (6.6) is too restrictive. It would be quite interesting but maybe challenging to prove whether the result in [6] is true or not for stochastic Schrödinger equations. In fact, as far as we know, people even do not know whether the results in [6, 16] are true or not for deterministic Schrödinger equations with time-dependent lower order term coefficients, which is a particular case of the equation (1.1).

**Acknowledgments.** This paper is an improved version of one chapter of the author's Ph D thesis ([18]) accomplished at Sichuan University under the guidance of Professor Xu Zhang. The author would like to take this opportunity to thank him deeply for his help. The author also highly appreciates the anonymous referees for their constructive comments.

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